The Mathematical Society of Japan

The 17th MSJ-SI

Modular Forms and Multiple Zeta Values

Abstracts

February 17 (Mon)

Francis Brown (University of Oxford)

Meromorphic modular forms and a conjecture of Gross-Zagier

This is joint work with Tiago J. Fonseca. Gross and Zagier formulated an algebraicity conjecture about the values of higher Greens functions at CM points on modular curves which has attracted a lot of interest. The goal of this project was to interpret the conjecture geometrically, and our main result states that if one of the standard conjectures about motives of cusp forms holds (namely 'conservativity'), then the conjecture is true automatically. However I will not talk about motives and will instead focus on two elementary aspects of this project. The first concerns meromorphic modular forms (i.e., modular forms with prescribed poles) which do not seem not to be very well-known and should have many applications. The second is a construction of matrix-valued Green's functions with remarkable properties. The latter suggests that the Gross-Zagier conjecture has a generalisation to non-CM points which involves special values of L-functions.

Chieh-Yu Chang (National Tsing Hua University)

On special v-adic gamma values after Gross-Koblitz-Thakur

In this talk, I will present my recent joint work with Fu-Tsun Wei and Jing Yu. I will introduce v-adic special gamma values over function fields in positive characteristic (defined by Goss and Thakur), which serve as analogues to the special values of Morita's p-adic gamma function. Our main result demonstrates that all algebraic relations among these special v-adic values come from the standard functional equations together with the Gross-Koblitz-Thakur formula. An outline of the proof will also be provided.

February 18 (Tue)

Herbert Gangl (Durham University)

On the depth reduction for multiple polylogarithms

We outline parts of the history of Zagier's Polylogarithm Conjecture and then give an overview of recent developments. The conjecture, very much in the spirit of Dirichlet's famous Class Number Formula, expresses special values of the Dedekind zeta function $\zeta_F(s)$ of a number field F at integer s=n in terms of suitable n-logarithm combinations with arguments in F. A crucial change in proof strategy by Goncharov and Rudenko starts with multiple polylogs (rather than with the classical ones) in maximal depth, followed by successive depth reduction, culminating in a proof for weight $n = 4$. Subsequently further inroads have been made for higher weight (work by, and partially joint with, Charlton, Radchenko, Rudenko).

Shuji Yamamoto (Keio University)

Discretization of integral representation of multiple zeta values

In 2024, Maesaka, Seki and Watanabe found an unexpected identity between the multiple harmonic sums (finite partial sums of the series expression of multiple zeta values) and a certain kind of Riemann sums (approximation of the iterated integral expression). After their discovery, some variants, generalizations and applications were obtained by several authors. In this talk, I will survey these results.

Michel Waldschmidt (Sorbonne University)

Linear independence of odd zeta values using Siegel's lemma, following Stéphane Fischler

Hermite's proof of the transcendence of e in 1872 and Lindemann's proof of the transcendence of π used explicit auxiliary functions produced by some of Hermite's interpolation integrals. One main ingredient in the solution by Gel'fond and Schneider of Hilbert's 7th problem in 1932 was the use of the Thue–Siegel Lemma, which proves the existence of a suitable auxiliary function without giving an explicit formula. The proof by Apéry in 1976 of the irrationality of $\zeta(3)$, and the proofs by Rivoal, Ball, Zudilin and others on lower bounds for the dimension of the $\mathbb{Q}$–space spanned by odd zeta values rest, on explicit constructions. Recently, Fischler showed how to deal with a variant of the Thue–Siegel Lemma, and how this improves some of the earlier results.

Don Zagier (MPIM Bonn and ICTP Trieste)

Number theory arising from topology

I would like to tell about several different (though related) objects in pure number theory that emerged from topology, and more specifically from work that I have been donig over the last 15 years with Stavros Garoufalidis on quantum invariant of knots and 3-manifolds. The first of these is a construction (carried out jointly with Frank Calegari) of units in cyclotomic extensions of arbitrary number fields, starting with an element of the Bloch group of the field; the second is the notion of "quantum modular forms" and later also "holomorphic quantum modular forms", the latter being a generalization of mock modular forms; and the third is the generalization of the so-called Habiro ring (also a purely number-theoretical object that arose originally in topology) to Habiro rings of number fields (joint also with Peter Scholze and Campbell Wheeler). No prior acquaintance with Bloch groups, quantum modular forms, or the Habiro ring will be assumed.

February 19 (Wed)

Toshiki Matsusaka (Kyushu University)

A unified approach to Rohrlich-type divisor sums

We propose a systematic method for analyzing Rohrlich-type divisor sums for arbitrary congruence subgroups $\Gamma_0(N)$. Our main theorem unifies various results from the literature, and its significance is illustrated through the following five applications: (1) the valence formula, (2) a natural generalization of classical Rohrlich's formula to level N, (3) an explicit version of the theorem by Bringmann-Kane-Löbrich-Ono-Rolen, (4) an extension of the generalized Rohrlich formula proposed by Bringmann-Kane, and (5) an alternative proof of the decomposition formula for twisted traces of CM values of weight 0 Eisenstein series. This is a joint work with Daeyeol Jeon, Soon-Yi Kang, and Chang Heon Kim.

Özlem Imamoglu (ETH Zurich)

A Lyapunov exponent associated to modular functions

We define and prove properties of a $\mathrm{GL}(2, \mathbf{Z})$ invariant function, a Lyapunov exponent attached to the modular function j, generalizing a function defined by Spalding and Veselov in the case of the constant function 1. Our results were motivated by conjectures of Kaneko about the values of j at real quadratic irrationalities. This is joint work with Paloma Bengoechea and Sebastian Herrero.

February 20 (Thu)

Ken Ono (University of Virginia)

Fun with quasimodular forms

Yoshinori Mizuno (Nagoya Institute of Technology)

Some applications of general genus characters

We give explicit formulas of several L-functions associated with general genus characters. In particular, yet another proof of an explicit formula of genus character L-function of quadratic orders will be given. Also, some applications will be mentioned, which include a generalization of the Hirzebruch-Zagier formula about class numbers of imaginary quadratic fields, congruences between Hirzebruch sums and class numbers of quadratic orders, and a proof of Hutchinson's conjecture on a generalization of the Gross-Zagier formula about singular moduli. These are joint works in part with Masanobu Kaneko and with Jigu Kim.

Yen-Tsung Chen (Pennsylvania State University)

On the partial derivatives of Drinfeld modular forms of arbitrary rank

Let f be a holomorphic modular form of positive weight for a congruence subgroup $\Gamma \subset \mathrm{SL}_2(\mathbb{Z})$. It is well known that the derivative of f is not a modular form, but a quasi-modular form. In this talk, our aim is to present results concerning partial derivatives of Drinfeld modular forms in arbitrary rank $r \geq 2$. In particular, we propose a potential analouge of Drinfeld quasi-modular forms for $\mathrm{GL}_r(\mathbb{F}_q[\theta])$ that specialize to Drinfeld quasi-modular forms in the rank 2 setting defined by Bosser and Pellarin. Some transcendence properties of their special values at CM points will also be discussed. This is joint work with Oğuz Gezmiş.

Federico Pellarin (Sapienza University of Rome)

On 'eulerian' multiple zeta values for curves

We discuss positive characteristic multiple zeta values associated to general curves over $\mathbb{F}_q$ together with an $\mathbb{F}_q$-rational point introduced by Thakur. For the case of the projective line these values were defined as analogues of classical multiple zeta values. We prove a conjecture of Lara Rodríguez and Thakur on such multiple zeta values that are proportional, up to an algebraic coefficient, to a power of a function field analogue of $2i\pi$. This is joint work with Tuan Ngo Dac and Kwun Chung.

February 21 (Fri)

Soon-Yi Kang (Kangwon National University)

Unveiling Patterns in Partitions, Modular Forms, and Others

This talk explores various patterns and structures arising in partition theory, mock & quasi-modular forms, and related topics. We will discuss combinatorial and number-theoretic perspectives on these objects, highlighting unexpected connections and applications.

Takeshi Ogasawara (Dokkyo Medical University)

Some topics on mod p modular forms of weight one

For the weight one modular forms, we have difficulties in the theory and computations. In this talk, l will present a simple method for computing weight one modular forms over finite fields under a certain condition for the level. As an application, we can find $\mathrm{PGL}(2, p)$ and $\mathrm{PSL}(2, p)$ number fields with prescribed ramifications. Also, I will talk about some observations we noticed in our computations.

Wadim Zudilin (Radboud University Nijmegen)

A basic basic hypergeometric identity

An investigation of supercongruences for truncated ${}_4F_3$ hypergeometric sums reveals an interesting structure for their q-counterparts at roots of unity. While the corresponding hypergeometric series are linked with the L-values of certain modular eigenforms of weight 4, their q-versions give one a strange example of a q- ("basic") hypergeometric sum: it does not possess a closed form at roots of unity but its quotients for different values of auxiliary parameter do. A translation of the closed forms into congruences for q-sums and specialisation $q \to 1$ lead to a criterion for deciding whether a given prime is ordinary for the associated modular form.

The talk is partly based on joint work with Christian Krattenthaler.

Masanobu Kaneko (Kyushu University)

Finite multiple zeta values and other elements in the "poor man's adele ring"

The ring we consider is the direct product of all finite prime fields modulo the direct sum. Finite multiple zeta values are living in this ring and have an amazingly rich structure comparable to the original multiple zeta values. After a brief introduction to their theory and the "main conjecture", we exhibit some other elements in this ring, like an analogue of Euler's constant. This is joint work with Don Zagier and in part with Toshiki Matsusaka and Shin-ichiro Seki.

February 22 (Sat)

Michael Hoffman (U. S. Naval Academy)

Truncated multiple zeta values and summation formulas

We define truncated multiple zeta values (tMZV's) by

$$\zeta_n(a_1,\dots,a_k) = \sum_{n\ge n_1>n_2>\cdots>n_k\ge 1} \frac{1}{n_1^{a_1} n_2^{a_2}\cdots n_k^{a_k}}$$

where the a_i are integers. Since the sum is finite, the a_i need not be positive. With 0 allowed in the argument string there are interesting identities: for example, $\zeta_n(0,a_1,\dots,a_k)$ can be interpreted as both $\sum_{j=1}^{n-1}\zeta_j(a_1,\dots,a_k)$ and as

$$n\zeta_n(a_1,\dots,a_k) - \zeta_n(a_1-1,a_2,\dots,a_k)$$

The tMZV's satisfy the same kind of quasi-shuffle algebraic identities as MZV's. By applying these to the harmonic numbers $H_n=\zeta_n(1)$ we can get summation formulas for terms of form $n^aH_n^b$. With them one can prove identities like

$$\sum_{n=1}^{\infty} H_n^3\left(\zeta(2)-\sum_{k=1}^{n}\frac{1}{k^2}-\frac{1}{n}\right) = -\frac{11}{2}\zeta(4)+\zeta(3)+3\zeta(2)-6.$$

There are also interesting summation formulas involving the alternating harmonic numbers $\zeta_n(\bar{1})=\sum_{j=1}^{n}\frac{(-1)^j}{j}$ and their powers.

Kentaro Ihara (Kindai University)
Multiple Dedekind zeta values

In this talk, we will define multiple Dedekind zeta functions associated with algebraic number fields and explain their basic properties. Using a sequence consisting of the number of ideals with a given norm, we will construct these functions in a straightforward way and explain their fundamental properties such as convergence, iterated integral representation, and shuffle product. We will also discuss the analytic continuation using the method given by Matsumoto-Tanigawa based on the Mellin-Barnes integral and show that number-theoretic quantities such as class numbers appear in special values at points with negative components. This research is a joint work with Kenta Matsuda.