The Stable Marriage Problem: A New Algorithm

A. A. K. Majumdar
College of Asia-Pacific Management, Ritsumeikan Asia-Pacific University
H. Gunarto
College of Asia-Pacific Studies, Ritsumeikan Asia-Pacific University

Abstract

In the current literature, the Gale-Shapley algorithm (GS-algorithm, for short) is the only one to find a stable marriage for given preference patterns of n boys and n girls. In this paper, we present a new algorithm to find a stable marriage. From the computational point of view, our algorithm seems to be at least as good as the GS-algorithm. Starting with a given initial marriage (for given preference patterns of the n boys and n girls), the GS-algorithm cannot be employed to reach a stable marriage, but the proposed algorithm can deal with such a case as well. Moreover, in our treatment of the problem, the proofs of some of the existing results become more simplified.

AMS (1991) Subject classification: 05A05, 06A12, 30D40.

Keywords: Marriage problem, stable marriage, Gale-Shapley algorithm.
Published @ The Chittagong University Journal of Science, Vol. 28, No 2, page 7-25, July 2004, A.A.K. Majumdar and Hary Gunarto, G.C. Ray.

1. Introduction

Let us consider a set of n boys and n girls. Each person ranks, without ties, those of the opposite sex in accordance with his or her preferences for a marriage partner. A *marriage* is a one-to-one mapping from the set of n boys onto the set of n girls. Gale and Shapley (1962) introduced the concept of *unstable pair* in a marriage, which is defined as a pair of a boy and a girl who are not married to each other but prefer each other to their actual marriage mates. A marriage is called *stable* if there is no unstable pair in it. Given the sets of n boys and n girls together with their respective preference patterns, the problem is to find a stable marriage.

Gale and Shapley have proved that, given the sets of n boys and n girls and their individual preferences for persons of opposite sex, there always exists a stable marriage by explicitly constructing such a stable marriage. The algorithm, commonly known as the Gale-Shapley algorithm (GS-algorithm, for short) in the literature, is as follows: Let each boy propose to his most favorite girl. Each of the girls receiving more than one proposal rejects all but her favorite. However, she does not accept him yet, and rather keeps him on a string so that she may accommodate someone better that might come along later. The boys rejected now propose to their second choices. Each girl receiving new proposals chooses her favorite from the new proposers as well as the boy on the string, if any. She rejects all the rest and again keeps her favorite on a string. Continuing in this fashion, the algorithm will terminate, in a finite number of stages, resulting in a stable marriage. The GS-algorithm terminates after at best $(n-1)^2+1$ stages.

The stable marriage resulting from the above GS-algorithm may be called the *boy-oriented stable marriage*. In the GS-algorithm, interchanging the roles of boys and girls, the *girl-oriented stable marriage* is obtained. In general, the boy-oriented stable marriage is different from the girl-oriented stable marriage.

The stable marriage problem as well as the GS-algorithm has been treated extensively by Knuth (1997), Gusfield and Irving (1989), and Roth and Sotomayor (1999), and to some extent by Lawler (2001). Knuth gives different versions of computer programs for the GS-algorithm.

The important results related to the marriage problem are given in Theorems 1.1-1.5.

Theorem 1.1 (Gale and Shapley, Knuth) : There always exists a stable marriage (in any complete system of preferences).

Theorem 1.2 (Gale and Shapley) : For given system of preferences, the boy-oriented (girl-oriented) GS-algorithm is boy-optimal (girl-optimal) in the sense that each boy (girl) in the resulting marriage is at least as well off as he would be in any other stable marriage.

The following result has been established by Dubins and Freedman (1981). Hwang (1989) gives a more simplified proof.

Theorem 1.3 (Dubins and Freedman, Hwang) : For given system of preferences, if several boys collude in a GS-algorithm, each using a false rank ordering, they cannot all get better girls, where “better” is relative to each boy’s true rank ordering.

Theorem 1.4 (Lawler) : For given system of preferences, if both the boy-oriented and girl-oriented GS-algorithms lead to the same stable marriage, then there is a single stable marriage.

Theorem 1.5 (Roth and Sotomayor) : For any given system of preferences, starting from an initial unstable marriage, there is a sequence of marriages which finally lead to a stable marriage.

In the next section, we give some results related to the stability of marriage for given system of preferences. Following Gale and Shapley, we use the numerical values to denote the ranks; thus, given a set of n boys and n girls, the rankings of each person for those of the opposite sex are given by the numbers 1 through n. Our algorithm is given in §3, followed by some examples to illustrate our algorithm in §4. We conclude this paper with some comments and observations in the final section.

2. Some Preliminary Results

Throughout this paper, the set of n boys would be denoted by $B=\{b_1, b_2,..., b_n\}$ and the set of n girls would be denoted by $G=\{g_1, g_2,..., g_n\}$.

Each boy $b_i \in B$ $(1\leq i\leq n)$ ranks, without ties, all the girls in G in accordance with his preferences as a marriage partner; let the ranking be denoted by the numbers from 1 to n, where rank 1 stands for the best choice, rank 2 for the second best choice, and so on, and rank n stands for the least choice. This results in the *preference matrix* $X_n=(x_{ij})$ for the boys in B, where x_{ij} denotes the ranking of the i-th boy, b_i, for the j-th girl, g_j, and hence

$$1\leq x_{ij}\leq n \text{ for all } 1\leq i,j\leq n;\ x_{ij} \neq x_{ik} \text{ if } j\neq k \text{ (for all } 1\leq i\leq n).$$

Similarly we have the preference matrix $Y_n=(y_{ij})$ for the girls in G, where y_{ij} denotes the ranking of the i-th girl, g_i, for the j-th boy, b_j, so that

$$1\le y_{ij}\le n \text{ for all } 1\le i,j\le n;\ y_{ij}\ne y_{ik} \text{ if } j\ne k \text{ (for all } 1\le i\le n).$$

The two preference matrices $X_n=(x_{ij})$ and $Y_n=(y_{ij})$ can be combined together into the following bimatrix P_n, called the *preference bimatrix* for the boys and the girls:

$$P_n=\begin{array}{c} \\ \mathbf{b_1} \\ \mathbf{b_2} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{b_i} \\ \cdot \\ \cdot \\ \cdot \\ \mathbf{b_n} \end{array}\begin{array}{c} \begin{array}{ccccccc} \mathbf{g_1} & \mathbf{g_2} & \cdots & \mathbf{g_j} & \cdots & \mathbf{g_n} \end{array} \\ \begin{pmatrix} (x_{11}, y_{11}) & (x_{12}, y_{21}) & \cdots & (x_{1j}, y_{j1}) & \cdots & (x_{1n}, y_{n1}) \\ (x_{21}, y_{12}) & (x_{22}, y_{22}) & \cdots & (x_{2j}, y_{j2}) & \cdots & (x_{2n}, y_{n2}) \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ (x_{i1}, y_{1i}) & (x_{i2}, y_{2i}) & \cdots & (x_{ij}, y_{ji}) & \cdots & (x_{in}, y_{ni}) \\ \cdot & & & & & \\ \cdot & & & & & \\ \cdot & & & & & \\ (x_{n1}, y_{1n}) & (x_{n2}, y_{2n}) & \cdots & (x_{nj}, y_{jn}) & \cdots & (x_{nn}, y_{nn}) \end{pmatrix} \end{array},$$

where, in the pair (x_{ij},y_{ji}), x_{ij} gives the ranking of the boy b_i for the girl g_j, and y_{ji} gives the ranking of the girl g_j for the boy b_i, for all $1\le i,j\le n$.

For any boy $b_i\in B$, if the girl g_j is preferred to the girl g_ℓ, we use the notation

$$g_j\underset{b_i}{\succ}g_\ell, \text{ (or equivalently, } g_\ell\underset{b_i}{\prec}g_j).$$

Similarly, the notation

$$b_i\underset{g_j}{\succ}b_k, \text{ (or equivalently, } b_k\underset{g_j}{\prec}b_i\text{)}$$

means that the girl g_j prefers the boy b_i to the boy b_k. We then have the following result.

Lemma 2.1 : The preference relation $\succ$ (or, $\prec$) is transitive, that is,

$$(1)\ g_j\underset{b_i}{\succ}g_\ell,\ g_\ell\underset{b_i}{\succ}g_q \Rightarrow g_j\underset{b_i}{\succ}g_q,\quad (2)\ b_i\underset{g_j}{\succ}b_k,\ b_k\underset{g_j}{\succ}b_p \Rightarrow b_i\underset{g_j}{\succ}b_p.$$

$$(\ g_j\underset{b_i}{\prec}g_\ell,\ g_\ell\underset{b_i}{\prec}g_q \Rightarrow g_j\underset{b_i}{\prec}g_q,\ b_i\underset{g_j}{\prec}b_k,\ b_k\underset{g_j}{\prec}b_p \Rightarrow b_i\underset{g_j}{\prec}b_p.)$$

Proof : We first note that

$$g_j\underset{b_i}{\succ}g_\ell \text{ if and only if } x_{ij}<x_{i\ell},\ b_i\underset{g_j}{\succ}b_k \text{ if and only if } y_{ji}<y_{jk}.$$

To prove part (1) of the lemma, let $g_j\underset{b_i}{\succ}g_\ell,\ g_\ell\underset{b_i}{\succ}g_q$. Then,

$$x_{ij}<x_{i\ell},\ x_{i\ell}<x_{iq} \Rightarrow x_{ij}<x_{iq} \Rightarrow g_j\underset{b_i}{\succ}g_q.$$

The proof of part (2) is similar and is omitted here.

The preference relation $\succ$ (or, $\prec$) forms a complete ordering on each of the sets B and G. Lemma 2.1 shows that the individuals in the sets B and G are *rational*.

A *marriage*, M, is a 1–1 mapping, M: B→G, and would be denoted by

$$M=\begin{pmatrix} b_1 & b_2 & \cdots & b_i & \cdots & b_n \\ g_j & g_j & \cdots & g_j & \cdots & g_j \end{pmatrix}$$

1 2 i n

where $g_{j_i} \neq g_{j_k}$ and $j_i \neq j_k$ if $i\neq k$, $j_i \in \{1,2,\dots,n\}$ for all $1\le i\le n$.

Given the preference bimatrix P_n, there are n! possible marriages. In the marriage M above, the pair (b_i,g_{j_i}) is called a *marriage partner* for all $1\le i\le n$. The marriage partner (b_i,g_{j_i}) is called *stable* if and only if none of the pairs (b_i,g_{j_ℓ}), $1\le \ell\le n$, $\ell\neq i$, is unstable, and none of the pairs (b_k,g_{j_i}), $1\le k\le n$, $k\neq i$, is unstable.

Given the preference bimatrix P_n, the problem is to find a stable marriage. Theorem 1.1 guarantees the existence of (at least) one stable marriage.

For a given preference bimatrix P_n, given the marriages

$$M = \begin{pmatrix} b_1 & b_2 & \cdots & b_i & \cdots & b_n \\ g_{j_1} & g_{j_2} & \cdots & g_{j_i} & \cdots & g_{j_n} \end{pmatrix},\quad M' = \begin{pmatrix} b_1 & b_2 & \cdots & b_i & \cdots & b_n \\ g_{\ell_1} & g_{\ell_2} & \cdots & g_{\ell_i} & \cdots & g_{\ell_n} \end{pmatrix},$$

boy b_i prefers M to M' if and only if $g_{j_i} \succ_{b_i} g_{\ell_i}$, that is, if and only if $x_{ij_i} < x_{i\ell_i}$, $(1\le i\le n)$.

The notation $M \succ_B M'$ means that all the boys like M at least as well as M', with at least one boy preferring M to M', so that

$$x_{ij_i} \le x_{i\ell_i} \text{ for all } 1\le i\le n \text{ with } x_{kj_k} < x_{k\ell_k} \text{ for at least one k.}$$

Similar definition applies for the notation $M \succ_G M'$.

In the following lemmas and corollaries, we deduce some results related to the stability of marriages.

Lemma 2.2 : Given a marriage (for a given preference bimatrix P_n)

$$M = \begin{pmatrix} \cdots & b_i & \cdots & b_k & \cdots \\ & g_j & & g_\ell & \end{pmatrix},$$

the pair (b_i,g_ℓ) is unstable if and only if (1) $x_{i\ell}<x_{ij}$, and (2) $y_{\ell i}<y_{\ell k}$.

Proof : The pair (b_i,g_ℓ) is unstable in the marriage M if and only if $g_\ell \succ_{b_i} g_j$, $b_i \succ_{g_\ell} b_k$, which give respectively the first and second inequalities of the lemma. ∎

The following result is a consequence of Lemma 2.2.

Corollary 2.1 : In the marriage (for a given preference bimatrix P_n)

$$M = \begin{pmatrix} \cdots & b_i & \cdots & b_k & \cdots \\ & g_j & & g_\ell & \end{pmatrix},$$

let the pair (b_i,g_ℓ) be not unstable. Then,

(1) $x_{i\ell}<x_{ij} \Rightarrow y_{\ell i}>y_{\ell k}$, (2) $y_{\ell i}<y_{\ell k} \Rightarrow x_{i\ell}>x_{ij}$.

Proof : Let, in the marriage M, the pair (b_i,g_ℓ) be not unstable.

To prove part (1), let $x_{i\ell}<x_{ij}$. Then, $y_{\ell i}>y_{\ell k}$, for otherwise, $y_{\ell i}<y_{\ell k}$, which together with the given condition implies that the pair (b_i,g_ℓ) is unstable, contradiction to the hypothesis.
The proof of part (2) is similar, and is omitted here. أ

Corollary 2.2 : In the marriage (for a given preference bimatrix P_n)

$$M=\begin{pmatrix} b_1 & b_2 & & b_i & & b_n \\ & & \dots & & \dots & \\ g_{j_1} & g_{j_2} & & g_{j_i} & & g_{j_n} \end{pmatrix},$$

let

$$x_{ij_i}=1 \text{ for all } 1\le i\le n;\ (y_{j_i i}=1 \text{ for all } 1\le i\le n).$$

Then, the marriage M is stable.

Proof : We consider the case when $x_{ij_i}=1$ for all $1\le i\le n$. The proof for the other case is similar and is omitted here.

Now, since

$$x_{ij_k}>x_{ij_i}=1 \text{ for any pair } (b_i,g_{j_k}),\ k\ne i,\ 1\le k\le n,$$

it follows that the pair (b_i,g_{j_k}) is not unstable for any $k\ne i$ and for any $1\le i\le n$. أ

Lemma 2.3 : Let, in the marriage (for a given preference bimatrix P_n)

$$M=\begin{pmatrix} & b_i & & b_k & & b_p & \\ \dots & & \dots & & \dots & & \dots \\ & g_j & & g_\ell & & g_q & \end{pmatrix},$$

both the pairs (b_p,g_j) and (b_p,g_ℓ) be unstable. Let $x_{pj}<x_{p\ell}$. Then, in the marriage

$$M'=\begin{pmatrix} & b_i & & b_k & & b_p & \\ \dots & & \dots & & \dots & & \dots \\ & g_q & & g_\ell & & g_j & \end{pmatrix},$$

(which is obtained from M by interchanging the girls g_j and g_q) none of the pairs (b_p,g_ℓ) and (b_i,g_j) is unstable. Moreover, if $y_{jp}=1$, then in the marriage M', none of the pairs (b_1,g_j), (b_2,g_j), . . ., (b_{p-1},g_j), (b_{p+1},g_j), . . ., (b_n,g_j) is unstable.

Proof : Since $x_{p\ell}>x_{pj}$, it follows that, in the marriage M', the pair (b_p,g_ℓ) is not unstable. Again, since in the marriage M, the pair (b_p,g_j) is unstable, it follows that $y_{jp}<y_{ji}$, which shows that in the marriage M', the pair (b_p,g_j) is not unstable. Finally, in the marriage M',

$$y_{jp}=1<y_{js} \text{ for all } s\ne p,\ 1\le s\le n.$$

Hence, the pairs (b_1,g_j), . . ., (b_{p-1},g_j), (b_{p+1},g_j), . . ., (b_n,g_n) cannot be unstable.
All these complete the proof of the lemma. أ

Note that under the given conditions of Lemma 2.3, in the marriage

$$M''=\begin{pmatrix} & b_i & & b_k & & b_p & \\ \dots & & \dots & & \dots & & \dots \\ & g_j & & g_q & & g_\ell & \end{pmatrix},$$

the pair (b_p,g_j) is unstable.

The generalization of the result of Lemma 2.3 is the following

Corollary 2.3: Let, in the marriage (for a given preference bimatrix P_n)

$$M=\begin{pmatrix} & b_{i_1} & & b_{i_2} & & b_{i_k} & & b_{i_p} & & b_s & \\ \dots & & \dots & & \dots & & \dots & & \dots & & \dots \\ & & & & & & & & & & \end{pmatrix},$$

$g_{j_1} \quad g_{j_2} \quad g_{j_k} \quad g_{j_p} \quad g_r$

the pairs (b_s,g_{j_1}), (b_s,g_{j_2}), ..., (b_s,g_{j_k}), ..., (b_s,g_{j_p}) be all unstable. Let

$$x_{sj_k} = \min\left\{x_{sj_1},\, x_{sj_2},\, \ldots,\, x_{sj_k},\, \ldots,\, x_{sj_p}\right\}.$$

Then, in the marriage

$$M' = \begin{pmatrix} & b_{i_1} & & b_{i_2} & & b_{i_k} & & b_{i_p} & & b_s & \\ \ldots & & \ldots & & \ldots & & \ldots & & \ldots & & \ldots \\ & g_{j_1} & & g_{j_2} & & g_r & & g_{j_p} & & g_{j_k} & \end{pmatrix},$$

(which is obtained from M by interchanging the girls g_r and g_{j_k}), none of the pairs

(b_s,g_{j_1}), (b_s,g_{j_2}), ..., $(b_s,g_{j_{k-1}})$, $(b_s,g_{j_{k+1}})$, ..., (b_s,g_{j_p}) and (b_i,g_{j_k}) is unstable.

Lemma 2.4 : For a given preference bimatrix P_n, no stable marriage can contain two marriage pairs in each of which both the boy and the girl have their last choices as marriage partners.

Proof: We consider the marriage

$$M = \begin{pmatrix} & b_i & & b_j & & b_p & \\ \ldots & & \ldots & & \ldots & & \ldots \\ & g_j & & g_k & & g_q & \end{pmatrix},$$

such that $x_{ij}=n=y_{ji}$, $x_{jk}=n=y_{kj}$. Then, clearly $x_{ik}<x_{ij}=n$, $y_{ki}<y_{kj}=n$, so that the pair (b_i,g_k) is unstable, and hence the marriage M is not stable.

An immediate consequence of Lemma 2.4 is the following

Corollary 2.4 : For a given preference bimatrix P_n, any stable marriage can contain at best one marriage pair in which both the boy and the girl have their last choices as marriage partners.

The following result is due to Hwang (1989).

Lemma 2.5 : For a given preference bimatrix P_n, let the marriage

$$M_n = \begin{pmatrix} & b_i & & b_j & & b_p & \\ \ldots & & \ldots & & \ldots & & \ldots \\ & g_j & & g_k & & g_q & \end{pmatrix}$$

be stable. Then, the marriage (submarriage)

$$M_{n-1} = \begin{pmatrix} & b_j & & b_p & \\ \ldots & & \ldots & & \ldots \\ & g_k & & g_q & \end{pmatrix},$$

obtained from M_n by deleting the marriage pair (b_i,g_j) is also stable with respect to the preference bimatrix P_{n-1}, where P_{n-1} is obtained from P_n by deleting its i-th row and j-th column. Also, if any three submarriages of M_n are stable (with respect to the appropriate preference bimatrix), then the marriage M_n itself is stable.

The following results are given in Knuth (1997), but we give here more simplified proofs.

Lemma 2.6 : Let, for a given preference bimatrix P_n,

$$M = \begin{pmatrix} & b_i & & b_k & & b_p & \\ \ldots & & \ldots & & \ldots & & \ldots \\ & g_j & & g_l & & g_q & \end{pmatrix}, \quad M' = \begin{pmatrix} & b_i & & b_k & & b_p & \\ \ldots & & \ldots & & \ldots & & \ldots \\ & g_q & & g_j & & g_\ell & \end{pmatrix}$$

be two stable marriages. Then, either

(1) $g_j \underset{b_i}{\succ} g_q$, $b_k \underset{g_j}{\succ} b_i$, or (2) $g_q \underset{b_i}{\succ} g_j$, $b_i \underset{g_j}{\succ} b_k$.

Proof : First, let $g_j \underset{b_i}{\succ} g_q$. Then, $x_{ij}<x_{iq}$. Now, since the marriage M′ is stable

$$x_{ij}<x_{iq} \Rightarrow y_{jk}<y_{ji} \Rightarrow b_k \underset{g_j}{\succ} b_i .$$

Next, let $g_q \underset{b_i}{\succ} g_j$, so that $x_{iq}<x_{ij}$ .Then, we have the following sequence of implications :

$$\begin{aligned} x_{iq}<x_{ij} &\Rightarrow y_{qp}<y_{qi} \text{ (since M is stable)} \\ &\Rightarrow x_{pq}>x_{p\ell} \text{ (since M′ is stable)} \\ &\Rightarrow y_{\ell p}>y_{\ell k} \text{ (since M is stable)} \\ &\Rightarrow x_{k\ell}>x_{kj} \text{ (since M′ is stable)} \\ &\Rightarrow y_{jk}>y_{ji} \text{ (since M is stable)} \end{aligned}$$

The last inequality shows that $b_i \underset{g_j}{\succ} b_k$.

All these complete the proof of the lemma. ∎

Corollary 2.5 : For a given preference bimatrix P_n, let the marriage

$$M=\begin{pmatrix} & b_i & & b_k & & b_p & \\ \cdots & & \cdots & & \cdots & & \cdots \\ & g_j & & g_\ell & & g_q & \end{pmatrix}$$

be stable with x_{ij}=n=yji. Then all stable marriages contain the pair (b_i,g_j).

Proof : If possible, let

$$M' = \begin{pmatrix} & b_i & & b_k & & b_p & \\ \cdots & & \cdots & & \cdots & & \cdots \\ & g_q & & g_j & & g_\ell & \end{pmatrix}$$

be another stable marriage, not containing the marriage pair (b_i,g_j).

Since $x_{ij}=n<x_{iq}$, it follows that $g_q \underset{b_i}{\succ} g_j$. Then, by part (2) of Lemma 2.4, $b_i \underset{g_j}{\succ} b_k$, which contradicts the fact that $y_{jk}<y_{ji}=n$.

This contradiction establishes the result. ∎

Corollary 2.6 : For a given preference bimatrix P_n, let the marriage

$$M=\begin{pmatrix} & b_i & & b_k & & b_p & \\ \cdots & & \cdots & & \cdots & & \cdots \\ & g_j & & g_\ell & & g_q & \end{pmatrix}$$

be stable with $x_{ij}=1=y_{ji}$. Then, any stable marriage contains the pair (b_i,g_j).

Proof : is by contradiction. So, let

$$M' = \begin{pmatrix} & b_i & & b_k & & b_p & \\ \cdots & & \cdots & & \cdots & & \cdots \\ & g_q & & g_j & & g_\ell & \end{pmatrix}$$

be another stable marriage, not containing the marriage pair (b_i,g_j). Now, since $x_{ij}=1<x_{iq}$, it follows that $g_j \underset{b_i}{\succ} g_q$. Therefore, by part (1) of Lemma 2.4, $b_k \underset{g_j}{\succ} b_i$, leading to the

contradiction that $y_{jk}>y_{ji}=1$. أ

The following result has been established by Gusfield and Irving (1989), but we give here a more simplified proof.

Corollary 2.7 : Let, for a given preference bimatrix P_n,

$$M=\begin{array}{c}\begin{array}{cccc} b_i & b_k & b_p & \end{array}\\ \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ g_j & g_l & g_q & \end{pmatrix}\end{array}, \quad M'=\begin{array}{c}\begin{array}{cccc} b_i & b_k & b_p & \end{array}\\ \begin{pmatrix} \cdots & \cdots & \cdots & \cdots \\ g_q & g_j & g_\ell & \end{pmatrix}\end{array}$$

be two stable marriages such that one of b_i and g_j prefers M to M'. Then, the other would prefer M' to M.

Proof : is immediate from Lemma 2.6. أ

3. A New Algorithm

The algorithm that we are going to propose starts with an initial marriage, $M^{(0)}$, where the most-preferred girl from among the available girls is assigned to each boy, starting with the boy b_1. The algorithm then checks, in the successive iterations, each boy for possible unstable pairs, starting with the boy b_n in the first iteration. Each iteration can be viewed as one consisting of divorce and remarriage, eliminating successively the unstable pairs. Thus, starting with the boy b_n, his most-preferred girl from among the girls forming unstable pairs with b_n, if any, is found out, and in the next iteration the marriage couple is formed with b_n and this girl. By Lemma 2.3 (and Corollary 2.3), b_n does not form an unstable pair with any of the remaining girls. If however, b_n, in the initial marriage $M^{(0)}$, does not form an unstable pair with any of the other girls, we check for the boy on the left, b_{n-1}, for possible unstable pairs with the girls other than his current marriage partner. If a divorce and a remarriage occur, the boy who got the marriage partner of the boy b_n in the initial marriage is checked for a possible unstable pair. This process continues and in a finite number of iterations, a stable marriage is obtained.

Given the preference bimatrix P_n, we construct the initial marriage

$$M^{(0)}=\begin{pmatrix} b_1 & b_2 & b_i & b_n \\ & & \cdots & \cdots \\ g_{j_1} & g_{j_2} & g_{j_i} & g_{j_n} \end{pmatrix}$$

as follows: Starting with the first boy, b_1, the most-preferred girl g_{j_1} is assigned to him (so that $x_{1j_1}=1$). The 1st row and j_1-th column of P_n are then crossed off. Next, to the second boy, b_2, the most-preferrd girl g_{j_2} is assigned from the available girls (so that $x_{2j_2} = 1$ or 2 with $x_{2j_2}=2$ if $x_{1j_1}=1$). Crossing off the 2nd row and j_2-th column of P_n, the most-preferred girl g_{j_3} is assigned to the third boy, b_3, from the available girls (so that $x_{3j_3}=1$ or 2 or 3). The process is continued with the remaining boys and girls, and finally, the girl left after (n–1) assignments is assigned to the boy b_n.

If the initial marriage $M^{(0)}$ is stable, then the problem is solved. Otherwise, starting with $M^{(0)}$, we have to find a stable marriage. To do so, the steps followed are as follows :

Step 1 : We encircle b_n and find a girl $g_{k_n} \in G - \{g_{j_n}\}$ such that the pair (b_n, g_{k_n}) is unstable. If there are more than one such girl, the tie is resolved by choosing $k_n\ (\neq j_n)$ such that

$$x_{nk_n} = \min\{ x_{nj_i} : x_{nj_i} < x_{nj_n}, j_i \neq j_n \}.$$

The girl g_{k_n} so determined is then put in a square-box, and a new marriage $M^{(1)}$ is formed by interchanging the girls g_{j_n} and g_{k_n}.

Step 2 : We encircle b_{k_n} and find his most-preferred girl $g_{\ell_n} \in G - \{g_{j_n}, g_{k_n}\}$ such that the pair (b_{k_n}, g_{ℓ_n}) is unstable. The girl g_{ℓ_n} is put into a square-box, and the girls g_{k_n} and g_{ℓ_n} are interchanged.

The process is continued.

If, at any stage of the process, we reach a boy, say, b_{p_n}, for which there is no unstable pair, that is, for each $j \neq j_n$, the pair (b_{p_n}, g_j) is stable, we encircle the first unchecked boy on the left of b_n and continue the process.

If, at some stage of the algorithm, a more-preferred girl is not available for a boy but at later stage that particular girl is forced to change her marriage partner, then it is necessary to check if this girl forms an unstable pair with the boy under consideration, in some subsequent iteration.

The algorithm presented here may be called the boy-oriented version; interchanging the roles of boys and girls in the algorithm, we have the girl-oriented version.

4. Some Examples

We illustrate our algorithm with the help of some examples.

Example 4.1: Let the preference bimatrix be

$$P_4 = \begin{array}{c} \\ \mathbf{b_1} \\ \mathbf{b_2} \\ \mathbf{b_3} \\ \mathbf{b_4} \end{array} \begin{array}{c} \begin{array}{cccc} \mathbf{g_1} & \mathbf{g_2} & \mathbf{g_3} & \mathbf{g_4} \end{array} \\ \begin{pmatrix} (3,1) & (2,3) & (1,4) & (4,1) \\ (2,2) & (1,4) & (3,1) & (4,3) \\ (3,4) & (2,2) & (1,3) & (4,4) \\ (1,3) & (3,1) & (2,2) & (4,2) \end{pmatrix} \end{array}.$$

The initial marriage, $M^{(0)}$, is constructed as follows : Since x_{31}=1, the girl g_3 is assigned to the boy b_1. Then the first row and third column of P_4 are crossed-off. In the resulting reduced matrix, x_{22}=1, and so the girl g_2 is assigned to the boy b_2. Next, the second row and second column of P_4 are crossed-off, and in the resulting 2x2 matrix, $x_{31}=3<4=x_{34}$, and so the girl g_1 is assigned to the boy b_3. Then, the remaining girl g_4 is assigned to the boy b_4. This gives the following initial marriage :

$$M^{(0)} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ g_3 & g_2 & \boxed{g_1} & g_4 \end{pmatrix}.$$

Next, the boy b_4 is encircled and we check b_4 for any possible unstable pairs. We note that since $x_{44}=4$, the boy b_4 prefers each of the remaining girls to his present marriage partner. Since $x_{41}=1$, $y_{14}=3<4=y_{13}$; $x_{43}=2$, $y_{34}=2<4=y_{31}$; $x_{42}=3$, $y_{24}=1<4=y_{22}$, it follows that, in $M^{(0)}$, all the three pairs (b_4,g_1), (b_4,g_3) and (b_4,g_2) are unstable. Since

$$\min_i \left\{ x_{4i} : i \neq 4 \right\} = x_{41} = 1,$$

the girl g_1 is put into a square-box, and in the next iteration, the girls g_1 and g_4 are interchanged. The new marriage after the first iteration is

$$M^{(1)} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ \boxed{g_3} & g_2 & g_4 & g_1 \end{pmatrix}.$$

The boy b_3 is encircled and then checked for possible unstable pairs. Since $x_{33}=1$, $y_{33}=2<4=y_{31}$, we see that g_3 is the most-preferred girl of b_3 such that (b_3,g_3) is unstable. The girl g_3 is put into a square-box, and in the next iteration, the girls g_3 and g_4 are interchanged. Thus, we get the new marriage

$$M^{(2)} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ g_4 & \boxed{g_2} & g_3 & g_1 \end{pmatrix}.$$

The boy b_1 is encircled and checked for possible unstable pair. In this case, though $x_{13}=1$ and g_3 is the most-preferred girl of b_1, by Lemma 2.3, the pair (b_1,g_3) is not unstable in the marriage $M^{(2)}$. Thus, for b_1, it is sufficient to check the girls g_2 (for which $x_{12}=2$) and g_3 (for which $x_{13}=3$) successively for possible unstable pairs. Since $y_{21}=3<4=y_{22}$, it follows that the pair (b_1,g_2) is unstable, and the girl g_2 is put into a square-box. In the next iteration, the girls g_2 and g_4 are interchanged, thereby giving the following new marriage :

$$M^{(3)} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ g_2 & g_4 & g_3 & \boxed{g_1} \end{pmatrix}.$$

In the marriage $M^{(3)}$, the boy to be checked for possible unstable pairs is b_2, which is encircled. It is sufficient to check the pairs (b_2,g_1) (for which $x_{21}=2$) and (b_2,g_3) (for which $x_{23}=3$), in turn, for possible unstable pairs. Since $y_{12}=2<3=y_{14}$, it follows that (b_2,g_1) is indeed unstable in $M^{(3)}$, and so the girl g_1 is put into a square-box. In the next iteration, interchanging the girls g_1 and g_4, the new marriage obtained is

$$M^{(4)} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ g_2 & g_1 & \boxed{g_3} & g_4 \end{pmatrix}.$$

Now, the boy to be checked for possible unstable pairs is b_4, which is encircled. By Lemma 2.3, the pair (b_4,g_1) is not unstable in the marriage $M^{(4)}$. Considering his next choice, g_3, we see that the pair (b_4,g_3) is unstable. The girl g_3 is put into a square-box. Interchanging the girls g_3 and g_4, the resulting new marriage is

$$M^{(5)} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ \boxed{g_2} & g_1 & g_4 & g_3 \end{pmatrix}.$$

Continuing in this way, we get, in successive iterations, the following marriages :

$$M^{(6)} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ g_4 & \boxed{g_1} & g_2 & g_3 \end{pmatrix}, \quad M^{(7)} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ g_1 & g_4 & g_2 & \boxed{g_3} \end{pmatrix},$$

$$M^{(8)} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ g_1 & g_3 & \boxed{g_2} & g_4 \end{pmatrix}, \quad M^{(9)} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ g_1 & g_3 & g_4 & g_2 \end{pmatrix}.$$

In the marriage $M^{(9)}$, the encircled boy b_3 does not have any unstable pair. Consequently, the marriage $M^{(9)}$ is the desired stable marriage.

To show that the marriage $M^{(9)}$ is indeed stable, we first note that, except for the last girl, g_4, in each iteration, a girl gets a more-preferred boy as her marriage partner, and hence, in successive iterations, except for the last girl g_4, the rankings of the corresponding marriage partners of all other girls either improve or remain the same. Now, we argue as follows : In the marriage $M^{(9)}$, the pair (b_3,g_2) (with $x_{32}=2$) is not unstable, by Lemma 2.3 ; the pair (b_3,g_3) (with $x_{33}=1$) is not unstable either, since (b_3,g_3) is not unstable in the marriage $M^{(5)}$ (by Lemma 2.3), and from the marriages $M^{(5)}$ and $M^{(7)}$,

$$b_4 \underset{g_3}{\succ} b_3,\ b_2 \underset{g_3}{\succ} b_4 \Rightarrow b_2 \underset{g_3}{\succ} b_3 .$$

Finally, in the marriage $M^{(9)}$, the marriage pairs (b_4,g_2), (b_2,g_3) and (b_1,g_1), determined at the 9-th, 7-th and 6-th iterations respectively, are clearly stable, the marriage pairs (b_4,g_2) and (b_2,g_3) are stable by Lemma 2.3, and the marriage pair (b_1,g_1) is stable because from the marriages $M^{(5)}$ and $M^{(8)}$,

$$b_3 \underset{g_2}{\succ} b_1,\ b_4 \underset{g_2}{\succ} b_3 \Rightarrow b_4 \underset{g_2}{\succ} b_1,$$

and from the marriages $M^{(1)}$, $M^{(4)}$ and $M^{(7)}$,

$$b_3 \underset{g_3}{\succ} b_1,\ b_4 \underset{g_3}{\succ} b_3,\ b_2 \underset{g_3}{\succ} b_4 \ \Rightarrow b_2 \underset{g_3}{\succ} b_1.$$

For this particular example, an alternative way of verifying the stability of the marriage $M^{(9)}$ is as follows : Since $y_{11}=1$, $y_{24}=1$, $y_{32}=1$, it follows that, in $M^{(9)}$, none of the three girls g_1, g_2 and g_3 can have unstable pairs with any of the boys other than her respective marriage partner. In particular, none of the pairs (b_3,g_3), (b_3,g_2) and (b_3,g_1) is unstable. Consequently, the marriage $M^{(9)}$ is stable.

In this case, the last girl is g_4, and the last boy, getting the last girl in the final stable marriage, is b_3. We note that, in finding the stable marriage, the preference pattern of the last girl, g_4, is immaterial, that is, for any preference pattern y_{4i}, $1 \le i \le n$ with $y_{4i} \ne y_{4k}$ if $i \ne k$, the stable marriage would be the one given in $M^{(9)}$.

In the girl-oriented version, the initial marriage is

$$M_G^{(0)} = \begin{pmatrix} g_1 & g_2 & g_3 & g_4 \\ b_1 & b_4 & b_2 & b_3 \end{pmatrix},$$

with $y_{11}=1$, $y_{24}=1$, $y_{32}=1$. Hence, none of the pairs (b_3,g_3), (b_3,g_2) and (b_3,g_1) is unstable. Consequently, the initial marriage $M_G^{(0)}$ is stable.

Thus, for the given preference bimatrix P_4, both the boy-oriented version and the girl-oriented version give the same stable marriage. Moreover, in the girl-oriented version, the stable marriage is obtained in a single iteration. It may be mentioned here that both the boy-oriented GS-algorithm and the girl-oriented GS-algorithm give the marriage $M^{(9)}$ as the stable marriage, and hence, by Theorem 1.4, for the given preference bimatrix P_4, there is only one stable marriage.

Example 4.2: For the preference bimatrix

$$P_4 = \begin{pmatrix} (4,1) & (2,4) & (1,4) & (3,3) \\ (4,2) & (3,1) & (1,2) & (2,4) \\ (4,4) & (2,2) & (1,3) & (3,1) \\ (2,3) & (1,3) & (3,1) & (4,2) \end{pmatrix},$$

the initial marriage is

$$M^{(0)} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ \boxed{g_3} & g_4 & g_2 & g_1 \end{pmatrix}$$

in which the boy b_4 is encircled for possible unstable pairs. Since $x_{42}=1$, $y_{24}=3>2=y_{23}$, it follows that the pair (b_4,g_2) is not unstable. So, we check b_3 for possible unstable pairs in $M^{(0)}$, and b_3 is encircled. In this case, $x_{33}=1$, $y_{33}=3<4=y_{31}$, and hence, the pair (b_3,g_3) is unstable in $M^{(0)}$, and the girl g_3 is put into a square-box. In the next iteration, the girls g_2 and g_3 are interchanged, and the new marriage is

$$M^{(1)} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ g_2 & g_4 & \boxed{g_3} & g_1 \end{pmatrix}.$$

In the marriage $M^{(1)}$, the boy b_1 is encircled and is to be checked for possible unstable pairs. However, since $x_{13}=1$, $y_{31}=4>3=y_{33}$, we see that the pair (b_1,g_3) is not unstable in the marriage $M^{(1)}$. So, we consider the boy b_2, since b_2 is the only one who has not yet been checked for possible unstable pairs. The boy b_2 is encircled, and since $x_{23}=1$, $y_{32}=2<3=y_{33}$, the pair (b_2,g_3) is unstable in the marriage $M^{(1)}$. The girl g_3 is put into a square-box, and in the next interation the girls g_3 and g_4 are interchanged. The resulting marriage is

$$M^{(2)} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ \boxed{g_2} & g_3 & g_4 & g_1 \end{pmatrix},$$

in which the boy b_3, encircled, is to be checked for possible unstable pairs. In this case, $x_{32}=2<3=x_{34}$, $y_{21}=4>2=y_{23}$, and so (b_3,g_2) is the (only) unstable pair in the marriage $M^{(2)}$. The girl g_2 is put into a square-box, and in the next interation, the girls g_2 and g_4 are interchanged to get the following new marriage

$$M^{(3)} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ g_4 & g_3 & g_2 & g_1 \end{pmatrix}.$$

The marriage $M^{(3)}$ is the desired stable marriage : In $M^{(3)}$, the boy b_2 has his first choice as the marriage partner ($x_{23}=1$), the pair (b_3,g_3) (with $x_{33}=1<2=x_{32}$) is not unstable (by virtue of Lemma 2.3, since (b_3,g_3) is not unstable in the marriage $M^{(2)}$), and finally, the pair (b_4,g_2) is not unstable (since (b_4,g_2) is not unstable in the initial marriage $M^{(0)}$).

In this case, the last girl is g_4 (not g_1 nor g_2), and the last boy is b_1 – the boy who got the last girl as his marriage partner in the final stable marriage.

In the girl-oriented version, the initial marriage is

$$M_G^{(0)} = \begin{bmatrix} g_1 & g_2 & g_3 & g_4 \\ b_1 & b_2 & b_4 & b_3 \end{bmatrix},$$

which is stable by Corollary 2.2, since $y_{11}=1$, $y_{22}=1$, $y_{34}=1$, $y_{43}=1$.

For this particular example, the boy-oriented version and the girl-oriented version of stable marriages are different. Moreover, it can be checked that the boy-oriented GS-algorithm gives the marriage $M^{(3)}$ as the stable marriage, and the girl-oriented GS-algorithm leads to the marriage $M_G^{(0)}$ as the stable marriage.

Example 4.3 : For the preference bimatrix $P_4= \begin{pmatrix} (1,4) & (4,3) & (2,4) & (3,4) \\ (4,3) & (1,4) & (2,3) & (3,2) \\ (1,1) & (2,2) & (3,1) & (4,3) \\ (3,2) & (2,1) & (1,2) & (4,1) \end{pmatrix}$, the

initial marriage is $M^{(0)}=\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ g_1 & \boxed{g_2} & g_3 & g_4 \end{pmatrix}$.

We now check for possible unstable pairs in the marriage $M^{(0)}$, starting from the boy b_4. The boy b_4 is encircled. Now, $x_{43}=1<4=x_{44}$, but $y_{34}=2>1=y_{33}$, so that the (b_4,g_3) is not unstable in $M^{(0)}$. However, since $x_{42}=2<4=x_{44}$, $y_{24}=1<4=y_{22}$, it follows that the pair (b_4,g_2) is unstable. The girl g_2 is put into a square-box, and in the next iteration, the girls g_4 and g_2 are interchanged. The new marriage is

$$M^{(1)}= \begin{pmatrix} b_1 & b_2 & \rightarrow b_3 & b_4 \\ \boxed{g_1} & g_4 & g_3 & g_2 \end{pmatrix}.$$

The boy b_2 is encircled. However, in the marriage $M^{(1)}$, $x_{23}=2<3=x_{24}$, but $y_{33}=1<3=y_{32}$, so that the pair (b_2,g_3) is not unstable. We therefore consider the boy b_3, encircle it and check for possible unstable pairs. Since $x_{31}=1<3=x_{33}$, $y_{13}=1<4=y_{11}$, the pair (b_3,g_1) is unstable. The girl g_1 is put into a square-box, and in the next iteration, the girls g_1 and g_3 are interchanged. The resulting new marriage is

$$M^{(2)}=\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ \boxed{g_3} & g_4 & g_1 & g_2 \end{bmatrix}.$$

The boy b_1 is encircled. In $M^{(2)}$, $x_{13}=2>1=x_{11}$, but the pair (b_1,g_1) is not unstable. We therefore move to the boy b_4 again, encircle it, and check if the pair (b_4,g_3) (which is not unstable in $M^{(0)}$) is unstable in $M^{(2)}$. Since $y_{34}=2<4=y_{31}$, we see that the pair (b_4,g_3) is indeed unstable. The girl g_3 is put into a square-box, and in the next iteration, the girls g_2 and g_3 are interchanged to get the following new marriage :

$$M^{(3)}=\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \\ g_2 & g_4 & g_1 & g_3 \end{bmatrix}.$$

The boy b_1 is encircled. But the pair (b_1,g_4) is not unstable in $M^{(3)}$, since $y_{41}=4>2=y_{42}$. So, we move to the boy b_2 again, encircle it, and check successively if the pairs (b_2,g_2) (for which $x_{22}=1$) and (b_2,g_3) (for which $x_{23}=2$ but is not unstable in $M^{(1)}$) is unstable in $M^{(3)}$. Since $y_{22}=4>3=y_{21}$, and $y_{32}=3>2=y_{34}$, it follows that none of the pairs (b_2,g_2) and (b_2,g_3) is unstable in $M^{(3)}$. Hence, the marriage $M^{(3)}$ is stable.

In the girl-oriented version, the initial marriage is

$$M_G^{(0)}=\begin{bmatrix} g_1 & g_2 & g_3 & g_4 \\ b_3 & \boxed{b_4} & b_2 & b_1 \end{bmatrix},$$

where the encircled girl, g_4, does not have an unstable pair with any of the boys b_4 (for which $y_{44}=1$), b_2 (for which $y_{42}=2$) and b_3 (for which $y_{43}=3$). So, we move to the girl g_3, and encircle it. Though the pair (g_3,b_3) (with $y_{33}=1$) is not unstable, the pair (g_3,b_4) (with $y_{34}=2$) is unstable. The boy b_4 is put into a square-box, and in the next iteration, the boys b_2 and b_4 are interchanged. The resulting new marriage is

$$M_G^{(1)}=\begin{bmatrix} g_1 & g_2 & g_3 & g_4 \\ b_3 & b_2 & b_4 & b_1 \end{bmatrix},$$

which is stable.

5. Discussion

Given the preference bimatrix P_n, our algorithm, presented in §3, leads to a stable marriage after a finite number of iterations. In the worst case, it takes $(n–1)^2$ iterations, as in Example 4.1.

It might be of some interest to compare our algorithm with the GS-algorithm. If, for a given the preference bimatrix P_n, there is only one stable marriage, then, of course, both the boy-oriented and girl-oriented versions of each of the two algorithms lead to that unique stable marriage, as in Example 4.1. However, in other cases where, for the given preference bimatrix P_n, there are more than one stable marriage, the situation may be different. Example 4.2 provides an instance where the boy-oriented version of our algorithm and the GS-algorithm lead to the same stable marriage, and the girl-oriented version of our algorithm and the GS-algorithm lead to another identical stable marriage. For the preference bimatrix given in Example 4.3, the boy-oriented version of our algorithm gives a stable marriage which is the same as that obtained by the girl-oriented version of the GS-algorithm, while the girl-oriented version of our algorithm and the boy-oriented version of the GS-algorithm both lead to another identical stable marriage. This example shows that Theorem 1.2 is not valid for the stable marriages obtained by our algorithm.

For the preference bimatrix $P_4^{(1)}=\begin{pmatrix} (4,1) & (3,4) & (1,1) & (2,3) \\ (3,4) & (4,1) & (2,2) & (1,2) \\ (4,2) & (3,2) & (1,3) & (2,1) \\ (4,3) & (2,3) & (3,4) & (1,4) \end{pmatrix}$, both the boy- and girl-oriented versions of our algorithm as well as the girl-oriented version of the GS-algorithm give the following stable marriage

$$M_{B(1)}=M_{G(1)}=M_{GGS(1)} = \begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ g_3 & g_2 & g_4 & g_1 \end{pmatrix},$$

while the boy-oriented version of the GS-algorithm leads to the stable marriage

$$M_{BGS(1)}=\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ g_3 & g_1 & g_4 & g_2 \end{pmatrix}.$$

Here, by our algorithm, the boy-oriented and girl-oriented versions require respectively 1 and 4 iterations to reach the stable marriage, $M_{B(1)}$, and the girl-oriented version of the GS-algorithm needs 2 iterations to reach the same stable marriage. Moreover, the boy-oriented version of the GS-algorithm leads to the corresponding stable marriage, $M_{BGS(1)}$, in 4 iterations. From this example, we see that Theorem 1.4 is not valid for our algorithm.

For the preference bimatrix $P_4^{(2)}=\begin{pmatrix} (3,1) & (4,2) & (2,1) & (1,3) \\ (3,4) & (4,3) & (1,2) & (2,2) \\ (2,3) & (4,1) & (1,4) & (3,1) \\ (2,2) & (1,4) & (4,3) & (3,4) \end{pmatrix}$, the boy-oriented versions of both the algorithms give the following stable marriage

$$M_{B(2)}=M_{BGS(2)}=\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ g_4 & g_3 & g_1 & g_2 \end{pmatrix};$$

by our algorithm, the initial marriage is stable, while, the GS-algorithm needs 1 iteration to give the same stable marriage. On the other hand, the girl-oriented versions of stable marriage are different by our algorithm and the GS-algorithm, given respectively by

$$M_{G(2)}=\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ g_3 & g_4 & g_1 & g_2 \end{pmatrix},\quad M_{GGS(2)}=\begin{pmatrix} b_1 & b_2 & b_3 & b_4 \\ g_3 & g_2 & g_4 & g_1 \end{pmatrix}.$$

To reach the corresponding stable marriages, the number of iterations required is 3 in each case.

For the preference bimatrix $P_4^{(1)}$, both the boy- and girl-oriented versions of our algorithm give the same stable marriage, but they are different by the GS-algorithm. However, for the preference bimatrix $P_4^{(2)}$, the boy- and girl-oriented versions of the two algorithms together give three stable marriages with

$$M_{B(2)} \underset{B}{\succ} M_{G(2)} \underset{B}{\succ} M_{GGS(2)},\quad M_{GGS(2)} \underset{G}{\succ} M_{G(2)} \underset{G}{\succ} M_{B(2)}.$$

We also compared our algorithm vis-a-vis the GS-algorithm in terms of the number of iterations required to reach the stable marriage, both for the boy-oriented and girl-oriented versions. For this, preference bimatrices were generated randomly. It has been observed that, so far as the number of iterations is involved, our algorithm is at least as good as the GS-algorithm for both the boy- and girl-oriented versions, and for most of the cases, our algorithm is more efficient, as shown in Fig. 5.1 below. A simplified version of our algorithm, not taking into consideration the space or time efficiency, may be found on the homepage of the second author (http://www.apu.ac.jp/~gunarto).

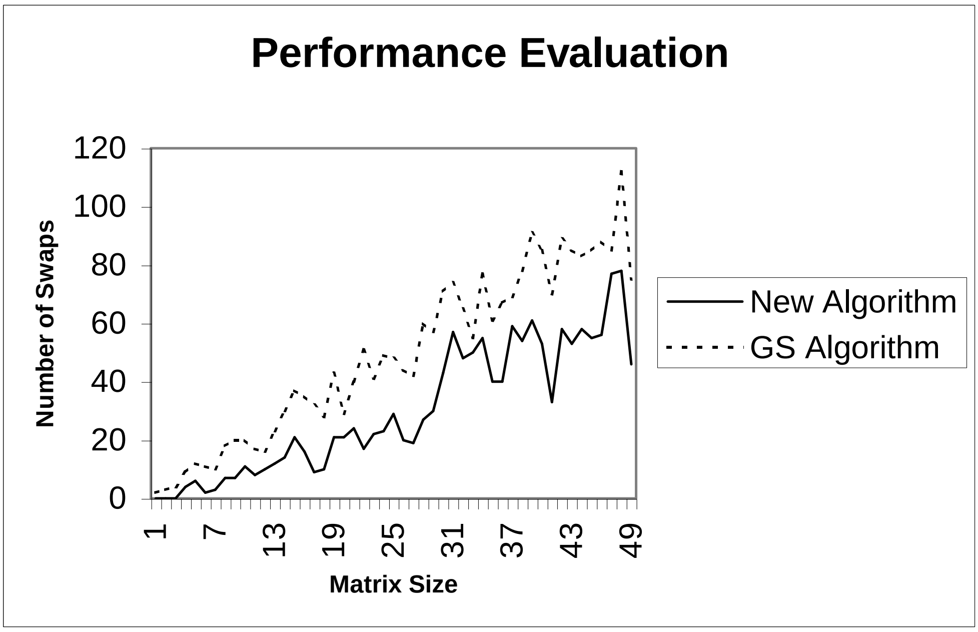

Figure 5.1. Number of Swaps vs. Matrix Size for Both Algorithms

Another advantage of our algorithm is that, starting from any unstable marriage for a given preference bimatrix P_n, the algorithm leads to a stable marriage. The GS-algorithm does not have any criteria to deal with such a problem. In his monograph, in Research Problem 8, Knuth raised the question: Can stability be always achieved by a sequence of divorces and remarriages, (like our algorithm)? Knuth gives an example which shows that sequential divorces and remarriages may lead to cycling, giving a negative answer to his question.

In the following example, we shall see how our algorithm leads to a stable marriage.

Example 5.1: Given the preference bimatrix $P_3 = \begin{pmatrix} (1,2) & (3,1) & (2,1) \\ (2,1) & (3,3) & (1,2) \\ (1,3) & (3,2) & (2,3) \end{pmatrix}$,

starting with the unstable marriage $M^{(0)} = \begin{pmatrix} b_1 & b_2 & b_3 \\ \boxed{} & g_1 & g_2 \end{pmatrix} \; g_3$

we get successively the following marriages:

$$M^{(1)} = \begin{pmatrix} b_1 & b_2 & b_3 \\ g_2 & g_1 & \boxed{g_3} \end{pmatrix}, \quad M^{(2)} = \begin{pmatrix} b_1 & b_2 & b_3 \\ g_2 & \boxed{g_3} & g_1 \end{pmatrix},$$

$$M^{(3)} = \begin{pmatrix} b_1 & b_2 & b_3 \\ g_3 & g_2 & \boxed{g_1} \end{pmatrix}, \quad M^{(4)} = \begin{pmatrix} b_1 & b_2 & b_3 \\ g_1 & g_2 & g_3 \end{pmatrix};$$

the marriage $M^{(1)}$ is obtained from $M^{(0)}$ by considering the unstable pair (b_2,g_1) and interchanging the girls g_1 and g_2, in $M^{(1)}$, the pair (b_2,g_3) is unstable and interchange of the girls g_1 and g_3 leads to the marriage $M^{(2)}$, where the pair (b_1,g_3) is unstable, and interchanging the girls g_2 and g_3, we get the marriage $M^{(3)}$ which is also unstable, and (b_1,g_1) is a unstable pair; however, interchanging the girls g_1 and g_3 give the marriage $M^{(4)}$, which is the same as the unstable marriage $M^{(0)}$ with which we started. Thus, cycling occurs, and we fail to reach to a stable marriage.

Let us now apply the criterion of our algorithm to the marriage

$$M^{(0)} = \begin{pmatrix} b_1 & b_2 & b_3 \\ g_1 & g_2 & \boxed{g_3} \end{pmatrix}.$$

we start with the boy b_3 and encircle it. But the pair (b_3,g_1) (with $x_{31}=1<2=x_{33}$) is not unstable. So, we move to the boy b_2 and encircle it. The pair (b_2,g_3) (with $x_{23}=1$) is unstable. So, we put g_3 into a square-box, and in the next iteration, the girls g_2 and g_3 are interchanged, giving the following marriage

$$M_1^{(1)} = \begin{pmatrix} b_1 & b_2 & b_3 \\ g_1 & g_3 & g_2 \end{pmatrix}.$$

In $M_1^{(1)}$, neither of the pairs (b_3,g_1) and (b_3,g_3) is unstable; also, since $x_{11}=1$, neither of the pairs (b_1,g_2) and (b_1,g_3) is unstable. Hence, the marriage $M_1^{(1)}$ is stable.

Thus, starting with any unstable marriage, our algorithm would always lead to a stable marriage. The question of arriving at a stable marriage, starting from any unstable marriage, has been answered in the affirmative by Roth and Sotomayor as well, but our approach is different from theirs.

It may be mentioned here that, for the given preference bimatrix P_3, the initial marriage according to our algorithm is stable.

In a recent paper, Wang (1998) offers a new algorithm for the stable marriage problem, but the criterion is different. Given the set of n boys and n girls as well as the preference bimatrix $P_n=(x_{ij},y_{ji})$, the objective is to find a marriage

$$M=\begin{pmatrix} & b_i & & b_j & & b_p & \\ \cdots & & \cdots & & \cdots & & \cdots \\ & g_j & & g_k & & g_q & \end{pmatrix},$$

such that the weight $W_{ij}=W_{ij}(x_{ij},y_{ji})$ is minimized.

Acknowledgement

The present work was done under a research grant of the Ritsumeikan Asia-Pacific University. The authors gratefully acknowledge the financial support of RAPU.

References

Dubins, L.E. and D.A. Freedman. 1981. “Machiavelli and the Gale-Shapley Algorithm”, *American Mathematical Monthly*, 88 : 485-494.

Gale, D. and L.S. Shapley. 1962. “College Admission and the Stability of Marriage”. *American Mathematical Monthly*, 69: 9-15.

Gusfield, Dan and Robert W. Irving. 1989. *The Stable Marriage Problem : Structure and Algorithms*. MIT Press Series in the Foundations of Computing, MIT Press.

Hwang, J. S. 1989. “On Classical and Modern Marriage Problems”. *International Journal of Computer Mathematics*, 28: 19-26.

Knuth, Donald E. 1997. *Stable Marriage and Its Relation to Other Combinatorial Problems : An Introduction to the Mathematical Analysis of Algorithms*. CRM Proceedings & Lecture Notes - Vol. 10. American Mathematical Society.

Lawler, E. L. 1976. *Combinatorial Optimization : Networks and Matroids*. Holt, Rinehart & Winston, New York.

Roth, Alvin E. and Marilda A. Oliveira Sotomayor. 1999. *Two-Sided Matching : A Study in Game-Theoretic Modeling and Analysis.* Cambridge University Press.

Wang, Mingwen. 1998. “A New Algorithm for Stable Assignment”, *International Journal of Computer Mathematics*, 66 : 1-7.