

Palindromes in Some Smarandache-Type Functions

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Abstrak. Target riset dan tujuan utama dari makalah ini adalah untuk menyelidiki sifat palindrom (urutan angka yang sama jika dibaca dari arah depan/maju maupun dari arah belakang/mundur) yang terdapat dalam tiga fungsi aritmatika tipe Smarandache; yaitu: fungsi Smarandache $S(n)$, fungsi pseudo Smarandache $Z(n)$, dan fungsi Sandor-Smarandache $SS(n)$.

Keywords:



*Smarandache function,
Sandor-Smarandache
function, palindromes.*

Abstract. The objective of this paper is to investigate for palindromes in three Smarandache type arithmetic functions, namely, the Smarandache function $S(n)$, the pseudo Smarandache function $Z(n)$, and the Sandor-Smarandache function $SS(n)$.

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1. Introduction

A palindromic number is one that reads the same forwards and backwards. Examples of palindromes are

$$1, 11, 101, 111, 121, \dots,$$

of which the first three are prime, the third and the fourth ones are composite. In general, in concatenation form, a palindrome N has one of the following two forms:

$$N = x_1 x_2 \dots x_n \dots x_2 x_1 \text{ or } N = x_1 x_2 \dots x_n x_n \dots x_2 x_1, \quad (1)$$

where $x_1 \in \{1, 2, \dots, 9\}$ and $x_k \in \{0, 1, 2, \dots, 9\}$ for all $2 \leq k \leq n$.

Possibly, the study of palindromic numbers in Smarandache-type arithmetical functions was initiated by Ashbacher [1], who made extensive search up to 100,000. He reports his findings, together with some unsolved questions and conjectures.

Ibstedt [2] has studied palindromic primes, and has given a list of all prime palindromes up to 9-digit numbers, found on computers. The following result is due to him.

Lemma 1.1. A palindrome with an even number of digits is a composite number.

Proof. A palindrome N with $2n$ digits has the form

$$N = x_1 x_2 \dots x_n x_n \dots x_2 x_1; x_1 \neq 0, 0 \leq x_i \leq 9 \text{ for each } i = 1, 2, \dots, n. \quad (2)$$

In decimal representation, N can be written as

$$N = x_1(10^{2n-1} + 1) + 10x_2(10^{2n-3} + 1) + 10^2x_3(10^{2n-5} + 1) + \dots + 10^{n-1}x_n(10 + 1). \quad (3)$$

Now, it can be proved by induction on k that $10^{2k-1} + 1$ is divisible by 11 for any integer $k \geq 1$. The proof is as follows: The result is clearly true for $k = 1$. So, we assume that $10^{2k-1} + 1$ is divisible by 11 for some integer k . Now, since

$$10^{2k+1} + 1 = 10^2(10^{2k-1} + 1) - (10^2 - 1) \quad (4)$$

it follows that the result is true for $k + 1$. This shows that N is divisible by 11.

A consequence of Lemma 1.1 is that, 11 is the only two-digit palindromic prime; moreover, higher digit palindromic primes must have odd number of digits, as has already been pointed out by Ibstedt.

Ibstedt [2] also introduced the concept of extended palindromes, which are palindromes of the form (1), where x_1, x_2, \dots, x_n are all natural numbers, of which at least one is greater than or equal to 10.

In this paper, we study the palindromic numbers in three commonly known Smarandache functions, namely, the Smarandache function $S(n)$, the pseudo Smarandache function $Z(n)$, and the Sandor-Smarandache function $SS(n)$. We further study the roles of the palindromic primes in the first two functions.

We consider the three cases separately in the next three sections. We conclude the paper with some remarks in the final Section 5.

2. Palindromes in $S(n)$

The Smarandache function, $S(\cdot) : \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$ (\mathbf{Z}^+ being the set of all positive integers) is defined by

$$S(n) = \min \{ m : m \geq 0, n \text{ divides } m! \}, n \geq 1, \quad (5)$$

with $S(1) = 1$.

Though explicit closed-form expressions for $S(n)$ are not available, we have the following two results (see, for example, Ashbacher [3]).

Lemma 2.1. Let

$$n = p_1^{\alpha_1} p_2^{\alpha_2} \dots p_k^{\alpha_k}$$

be the (unique) representation of n in terms of its prime factors p_1, p_2, \dots, p_k . Then,

$$S(n) = \max \left\{ S(p_1^{\alpha_1}), S(p_2^{\alpha_2}), \dots, S(p_k^{\alpha_k}) \right\}. \quad (6)$$

Lemma 2.2. For any prime $p \geq 2$, and any integer $k \geq 1$,

$$S(p^k) = \alpha p \text{ for some integer } \alpha \geq 1; \quad (7)$$

with

$$S(p) = p, S(p^p) = p^2. \quad (8)$$

In view of the above two lemmas, we see that, in order to search for palindromic numbers n for which $S(n)$ is also palindromic, it is sufficient to restrict our attention to palindromic primes whose powers and multiples are also palindromic. For example,

$$S(11k) = 11, S(101k) = 101 \text{ for all } 1 \leq k \leq 9, \quad (9)$$

$$S(k \cdot 11^2) = 22, S(k \cdot 11^3) = 33 \text{ for } 1 \leq k \leq 3,$$

$$S(k \cdot 101^2) = 202, S(k \cdot 101^3) = 303 \text{ for } 1 \leq k \leq 3,$$

$$S(101^4) = 404.$$

Now, given any integer $m \geq 1$, it is always possible to find a number n , for example, $n = m!$, such that $S(n) = m$. Thus, the function $S(\cdot)$ is onto. Clearly, $S(\cdot)$ is not a bijective, since, for example, $S(3) = 3 = S(6)$. Let $S^{(k)}(n)$ be the k -fold composition of $S(n)$ with itself, that is,

$$S^{(k)}(n) = \underbrace{(S \circ S \circ \dots \circ S)}_{k \text{ factors}}(n). \quad (10)$$

Note that, if p is a palindromic prime then for any integer n with $S(n) < p$ such that np is also a palindrome, we have

$$S(np) = p. \quad (11)$$

In fact, in such a case, for any integer $k \geq 1$,

$$S^{(k)}(np) = p. \quad (12)$$

The reader is referred to Liu [4] for a brief review of $S(n)$ till 2015. More results are given in Majumdar [5, 6]. A list of values of $S(n)$ for $n = 1, 2, \dots, 4800$ is given in Ibstedt [7]. We have made use of the values listed in [7] in some of the examples above.

3. Palindromes in $Z(n)$

The pseudo Smarandache function, $Z(\cdot) : \mathbf{Z}^+ \rightarrow \mathbf{Z}^+$, introduced by Kashihara [8], is defined by

$$Z(n) = \min \left\{ m : m \geq 1, n \text{ divides } \frac{m(m+1)}{2} \right\}, n \geq 1. \quad (13)$$

Explicit expressions for $Z(n)$ are available only for certain cases. A brief review of $Z(n)$ till 2015 is given in Liu [9]. The following results are due to Majumdar [5]; in some cases, only the relevant parts are given.

Lemma 3.1. If $p \geq 3$ is a prime and $n \geq 1$ is an integer, then

$$Z(np) = \begin{cases} p-1, & \text{if } 2n \text{ divides } p-1 \\ p, & \text{if } 2n \text{ divides } p+1 \end{cases}$$

Lemma 3.2. If $p \geq 3$ is a prime and n is an integer not divisible by p , then

$$Z(np^2) = \begin{cases} p^2 - 1, & \text{if } 2n \text{ divides } (p^2 - 1) \\ p^2, & \text{if } 2n \text{ divides } (p^2 + 1) \end{cases}$$

Lemma 3.3. Let $p \geq 3$ be a prime such that 4 divides $p + 1$. Then, $Z(2p^k) = p^k$ for any odd integer $k \geq 3$.

Lemma 3.4. Let $p \geq 5$ be a prime such that 3 divides $p + 1$. Then, $Z(3p^k) = p^k$ for any odd integer $k \geq 3$.

Lemma 3.5. Let $p \geq 3$ be a prime such that 8 divides $3p + 1$. Then, $Z(4p) = 3p$.

Lemma 3.6. Let $p \geq 5$ be a prime such that 5 divides $2p + 1$. Then, $Z(5p) = 2p$.

Lemma 3.7. Let $p \geq 3$ be a prime. Then,

$$Z(6p) = \begin{cases} p, & \text{if } 12 \text{ divides } p + 1 \\ 3p, & \text{if } 4 \text{ divides } 3p + 1 \end{cases}$$

Lemma 3.8. If $p \geq 3$ with $p \neq 7$ is a prime, then

$$Z(7p) = \begin{cases} 2p, & \text{if } 7 \text{ divides } 2p + 1 \\ 3p, & \text{if } 7 \text{ divides } 3p + 1 \end{cases}$$

Lemma 3.9. For any prime $p \geq 3$,

$$Z(8p) = \begin{cases} 3p, & \text{if } 16 \text{ divides } 3p + 1 \\ 5p, & \text{if } 16 \text{ divides } 5p + 1 \\ 7p, & \text{if } 16 \text{ divides } 7p + 1 \end{cases}$$

Lemma 3.10. For any prime $p \geq 5$,

$$Z(9p) = \begin{cases} p - 1, & \text{if } 18 \text{ divides } (p - 1) \\ p, & \text{if } 18 \text{ divides } (p + 1) \\ 2p - 1, & \text{if } 9 \text{ divides } (2p - 1) \\ 2p, & \text{if } 9 \text{ divides } (2p + 1) \\ 4p - 1, & \text{if } 9 \text{ divides } (4p - 1) \\ 4p, & \text{if } 9 \text{ divides } (4p + 1) \end{cases}$$

Lemma 3.11. For any prime $p \geq 3$ with $p \neq 5$,

$$Z(10p) = \begin{cases} 3p, & \text{if } 20 \text{ divides } p + 7 \\ 4p, & \text{if } 20 \text{ divides } p + 9 \\ 5p, & \text{if } 20 \text{ divides } p - 3 \end{cases}$$

Lemma 3.12. For any prime $p \geq 3$ with $p \neq 11$,

$$Z(11p) = \begin{cases} 2p, & \text{if } 11 \text{ divides } 2p + 1 \\ 3p, & \text{if } 11 \text{ divides } 3p + 1 \\ 4p, & \text{if } 11 \text{ divides } 4p + 1 \\ 5p, & \text{if } 11 \text{ divides } 5p + 1 \end{cases}$$

Lemma 3.13. If $p \geq 3$ is a prime, then

$$Z(12p) = \begin{cases} 3p, & \text{if } 8 \text{ divides } 3p + 1 \\ 7p, & \text{if } 24 \text{ divides } 7p + 1 \end{cases}$$

Lemma 3.14. For any prime $p \geq 2$,

$$Z(13p) = \begin{cases} 2p, & \text{if } 13 \text{ divides } 2p+1 \\ 3p, & \text{if } 13 \text{ divides } 3p+1 \\ 4p, & \text{if } 13 \text{ divides } 4p+1 \\ 5p, & \text{if } 13 \text{ divides } 5p+1 \\ 6p, & \text{if } 13 \text{ divides } 6p+1 \end{cases}$$

From Lemma 3.1 – Lemma 3.14, we see that, if p is a palindromic prime, then we may find an integer n such that np is also a palindrome with $Z(np) = kp$, where $k \geq 1$ is an integer such that kp is also a palindrome. Thus, for example, if $p = 11$ (which is the only 2–digit palindromic prime), then by virtue of Lemma 3.1, Lemma 3.7 and Lemma 3.10, each of $Z(22)$, $Z(33)$, $Z(66)$ and $Z(99)$ is palindrome with

$$Z(22) = 11 = Z(33) = Z(66), Z(99) = 44.$$

Now, $11^3 = 1331$ is a palindrome, and so by virtue of Lemma 3.3 and Lemma 3.4, both $Z(2.11^3) = Z(2662)$ and $Z(3.11^3) = Z(3993)$ are palindromes with

$$Z(2662) = 1331 = Z(3993).$$

Again, with the smallest 3–digit palindromic prime $p = 101$, we can form several palindromic $Z(n)$. By Lemma 3.1, Lemma 3.5, Lemma 3.7, Lemma 3.8, Lemma 3.9, Lemma 3.10 and Lemma 3.12, each of $Z(303)$, $Z(404)$, $Z(606)$, $Z(707)$, $Z(808)$, $Z(909)$ and $Z(1111)$ is a palindrome with

$$Z(303) = 101, Z(404) = 303 = Z(606) = Z(808),$$

$$Z(707) = 202, Z(909) = 404, Z(1111) = 505.$$

Since $101^3 = 1030301$ is also a palindrome, by Lemma 3.3 and Lemma 3.4,

$$Z(2.101^3) = Z(2060602) = 1030301 = Z(3090903) = Z(3.101^3).$$

Now, given any integer $m \geq 1$, it is always possible to find a number n , for example, $n = \frac{m(m+1)}{2}$ such that $Z(n) = m$. Thus, the function $Z(\cdot)$ is onto. However, $Z(\cdot)$ is not bijective, since, for example, $Z(2) = 3 = Z(6)$. Let $Z^{(k)}(n)$ be the k –fold composition of $Z(n)$ with itself, that is,

$$Z^{(k)}(n) = (\underbrace{Z \circ Z \circ \dots \circ Z}_{k \text{ factors}})(n). \tag{14}$$

The following question has been raised by Ashbacher [1]: Given a palindromic number n , what is the maximum number of times one can perform the functional compositions

$$Z(n), Z^{(2)}(n), \dots, Z^{(k)}(n), \dots,$$

and obtain a palindrome each time?

Ashbacher [1], using a computer program, searched for palindromic numbers n such that $Z(n)$ are also palindromes, in the range $10 \leq n \leq 10000$, and found that, of the 189 palindromic n , $Z(n)$ was palindromic only for 37 values of n . Over the same range, there is only one n , namely, $n = 909$, with $Z(n)$, $Z^{(2)}(n)$ and $Z^{(3)}(n)$ all palindromes:

$$Z(909) = 404, Z^{(2)}(909) = Z(404) = 303, Z^{(3)}(909) = Z(303) = 101.$$

Extending the range to $10 \leq n \leq 100000$, only one $n = 86868$ was found such that $Z(n)$, $Z^{(2)}(n)$, $Z^{(3)}(n)$ and $Z^{(4)}(n)$ all palindromes, with

$$Z(86868) = 17271, Z^{(2)}(86868) = Z(17271) = 2222,$$

$$Z^{(3)}(86868) = Z(2222) = 1111, Z^{(4)}(86868) = Z(1111) = 505.$$

It is interesting to observe here that, in the above case, all the palindromes, except 86868, are multiples of the palindromic prime 101:

$$1111 = 11 \times 101, 17271 = 3^3 \times 19 \times 101, 86868 = 2^2 \times 3^2 \times 19 \times 127.$$

The number 86868 shows that palindromes can result from non-palindromic primes as well.

For the reader who is interested in more detail on the pseudo Smarandache function, $Z(n)$, we refer to Majumdar [5, 6]. A list of the values of $Z(n)$ for $n = 1, 2, \dots, 5000$ is given in Gunarto and Majumdar [10]. Some of the values of $Z(n)$, given above, are taken from [10].

Another problem of interest is as follows: Is there any palindrome n such that $S(n) = Z(n)$? To answer the question, we first state the following result, a proof of which may be found in Majumdar [5].

Lemma 3.15. The solution of the equation $S(n) = Z(n)$ is $n = tp$, where p is a prime and t is an integer such that t divides $(p-1)!$, and $Z(tp) = p$.

Trivial examples are given below:

$$S(22) = 11 = Z(22), S(33) = 11 = Z(33), S(66) = 11 = Z(66).$$

By virtue of Lemma 3.1, we see that

$$S(2p) = p = Z(2p) \text{ for any odd prime } p \text{ such that } 4 \text{ divides } p + 1,$$

$$S(3p) = p = Z(3p) \text{ for any prime } p (\geq 5) \text{ such that } 4 \text{ divides } p + 1.$$

Thus, for example,

$$S(262) = 131 = Z(262), S(393) = 131 = Z(393).$$

4. Sandor-Smarandache Function

The Smarandache-type arithmetic function, denoted by $SS(n)$, posed by Sandor [11], and called the Sandor-Smarandache function, is defined as follows:

$$SS(n) = \max \left\{ k : 1 \leq k \leq n-1, n \text{ divides } \binom{n}{k} \right\}, n \geq 5 \quad (15)$$

where by convention,

$$SS(1) = 1, SS(2) = 1, SS(3) = 1, SS(4) = 1, SS(6) = 1.$$

The following result is due to Sandor [11].

Lemma 4.1. If $n (\geq 3)$ is an odd integer, then $SS(n) = n - 2$.

It can, in fact, be proved that $SS(n) = n - 2$ if and only if n is an odd integer. Some results related to the Sandor-Smarandache function are given in Majumdar [6]. The function was later studied in more detail by Islam and Majumdar [12], Islam, Gunarto and Majumdar [13], and more recently by [14] and [15]. The following result is also known about the Sandor-Smarandache function.

Lemma 4.2. $SS(n) = n - 3$ if and only if n is an even integer not divisible by 3.

We now prove the following result in connection with palindromes in $SS(n)$.

Lemma 4.3. There is an infinite number of integers n such that both n and $SS(n)$ are palindromic numbers.

Proof : Consider the palindromic number

$$\underbrace{99 \dots 9}_{k \text{ factors}} = 9(10^{k-1} + 10^{k-2} + \dots + 1); k \geq 2. \quad (16)$$

Since the sum on the right-hand side is $10^k - 1$, it follows that the integer n , defined by

$$n \equiv n(k) = (10^k - 1) + 2 = 10^k + 1,$$

is palindromic and odd. Therefore, by Lemma 4.1,

$$\underbrace{SS(n)}_{k \text{ factors}} = \underbrace{99 \dots 9}_{k \text{ factors}}; k \geq 2.$$

Thus, for example, from Lemma 4.3, we see that,

$$SS(101) = 99, SS(1001) = 999, SS(10001) = 9999, \dots$$

So far, these are the only known examples where both n and $SS(n)$ are palindromes. Note that, 101 is prime, 1001 is composite (which, by virtue of Lemma 1.1, is divisible by 11), 10001 is prime, 100001 is composite, and so on.

Let $SS^{(k)}(n)$ be the k -fold composition of $SS(n)$ with itself, that is,

$$SS^{(k)}(n) = (\underbrace{SS \circ SS \circ \dots \circ SS}_{k \text{ factors}})(n).$$

Then, we have the following lemma.

Lemma 4.4. For any integer $n (\geq 1)$,

$$SS^{(k)}(2n + 1) = 2(n - k) + 1 \text{ for any } 1 \leq k \leq n.$$

Proof : The proof is by induction on k . The result is clearly true for $k = 1$ (by virtue of Lemma 4.1). So, we assume the validity of the result for some k . But then

$$SS^{(k+1)}(2n + 1) = SS(SS^{(k)}(2n + 1)) = SS(2(n - k) + 1) = 2(n - k) - 1, \quad (17)$$

which shows the validity of the result for $k + 1$.

Lemma 4.4 shows that, it is possible to find a palindrome n such that $SS(n)$ is also a palindrome, by choosing an appropriate k . Thus, for example, $SS^{(2)}(101) = 99$ is a palindrome.

5. Some remarks

Starting with the palindromic prime 101, we observe that $101k$ is an extended palindrome for any $k = 10, 12, 13, \dots, 98$, with

$$S(101k) = 101.$$

But what happens if we apply $Z(\cdot)$ on these numbers? From Lemma 3.11, we see that $Z(1010)$ is not a palindrome. Lemma 3.13 shows that $Z(1212) = 303$ is a palindrome, but by Lemma 3.14, $Z(1313)$ is not a palindrome. A limited search shows that, $Z(1515)$, $Z(1717)$, $Z(1818)$, $Z(1919)$, $Z(2323)$, $Z(2424)$, $Z(2727)$, $Z(2929)$, $Z(3030)$, $Z(3838)$, $Z(3939)$ and $Z(4545)$ are all palindromes, with

$$Z(1515) = 404 = Z(1818) = Z(2727) = Z(3030) = Z(4545),$$

$$Z(1717) = 101, Z(1919) = 303 = Z(2424) = Z(3838),$$

$$Z(2323) = 505, Z(2929) = 202, Z(3939) = 909,$$

and $Z(3232)$, $Z(4343)$, $Z(4646)$, $Z(4747)$, $Z(4848)$ and $Z(4949)$ are extended palindromes, with

$$Z(3232) = 1919 = Z(4848), Z(4343) = 2020 = Z(4747),$$

$$Z(4646) = 2323, Z(4949) = 1616.$$

From $Z(9229) = 3355$, we see that n may be a palindrome with $Z(n)$ being an extended palindrome.

Earls [16] has introduced the concept of the recursive palindromic Smarandache values (RPSV) : An RPSV is a natural number n such that $S(n)$ is a palindrome, and deleting successively the rightmost digits of n and applying $S(\cdot)$ at each step gives a palindrome. An example of an RPSV is $n = 1514384$, with

$$S(1514384) = 94649, S(151438) = 373, S(15143) = 797, S(1514) = 757, S(151) = 151.$$

It turns out that the RPSVs are finite with $n = 1514384$ being the largest such number. Earls [16] then raises the following question:

Open Problem 5.1: What is the sequence of RPSVs when the leftmost digits are repeatedly deleted? Is the resulting sequence finite?

In connection with the Sandor-Smarandache function $SS(n)$, we have shown in Lemma 4.1 how to construct a palindromic integer n such that $SS(n)$ is also a palindrome. Now, we state the following open problem.

Open Problem 5.2: Is there any other palindromic number n such that $SS(n)$ is also a palindrome?

The proof of Lemma 4.3 gives the explicit form of the integer n such that both n and $SS(n)$ are palindromes. Note that, in view of Lemma 4.1, this is the only possible case when n is odd. Also, in view of Lemma 4.1 and Lemma 4.2, the following problem remains open.

Open Problem 5.3: Find $SS(n)$ when n is even and divisible by 3, that is, n is of the form $n = 6m$.

6. Conclusion

For some Smarandache-type functions $S(\cdot)$, $Z(\cdot)$, $SS(\cdot)$, our conclusion is as follows. In Smarandache function $S(\cdot)$, if p is a palindromic prime then for any integer n with $S(n) < p$ then np is also a palindrome, or $S(np) = p$ (Lemma 2.2).

For the pseudo Smarandache function $Z(\cdot)$, if p is a palindromic prime, then we may find an integer n such that np is a palindrome and $Z(np) = kp$ is also a palindrome, where $k \geq 1$. For example, if $p = 11$ (2-digit palindromic prime), then $Z(22)$, $Z(33)$, $Z(66)$ and $Z(99)$ is also palindrome i.e. $Z(22) = Z(33) = Z(66) = 11$, and $Z(99) = 44$ (Lemma 3.1 – Lemma 3.14).

And finally for the Sandor-Smarandache function $SS(\cdot)$, it is possible to find a palindrome n such that $SS(n)$ is also a palindrome, by choosing an appropriate k factors. (Lemma 4.4).

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